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On a stagnation condition for combining or branching inviscid flows

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A unique stagnation condition is derived as a guide for analysing systems of combined or branched flows through the use of the free-streamline theory and the Schwarz–Christoffel transformation. Moreover, the physical significance of such a condition is illustrated through its application to the analysis of the penetration and deflexion of jets normal to a bounded stream.

1. Introduction

Combining or branching flows in channels with or without free surfaces offer considerable complications (in fact, topological ambiguities), which are not easily surmounted. Ordinarily, the specification of the directions of flow in all branches, the positions of all stagnation points, and the velocity of one point in the channel (usually the upstream velocity in one of the channels) is sufficient to define the flow uniquely. If the velocities or the flow rates are prescribed at more points, then the stagnation points must be free to assume their appropriate positions, otherwise the flow will be overdetermined. In viscous flows in channels with properly rounded wedges, the uniqueness of the flow is maintained by the fact that the stagnation points tend to move to the points of maximum curvature on the wedges under the action of viscosity. If there are sharp corners asymmetrically placed in the flow, the stagnation points may assume positions which lead to separation and give rise to various types of flows such as Mestschersky-type flows, Kirchhoff-type flows, etc. (see e.g. Robertson 1965, p. 532). We must, therefore, assume that the stagnation points coincide with the sharp corners. This behaviour is illustrated in figure 1, which depicts both the physical plane and the transformation planes for a jet penetrating into a bounded stream. Clearly, the specification of velocities in both channels, the channel geometry, and the stagnation point render the flow overdetermined. This results from the fact that the common streamline BE may approach the stagnation point B in an infinite variety of directions. If EFG corresponds to the dividing streamline on the Ω plane (figure 1 (*b*)), the u component of velocity will vanish before v if the stagnation point B is approached. Similarly, if EHK corresponds to the dividing streamline, v will vanish before u, leading to a velocity or pressure discontinuity in the vicinity of the stagnation point. Thus a stagnation condition has to be specified so that u and v become identical and vanish simultaneously as R approaches B, i.e. the common streamline bisects the wedge at B. Such

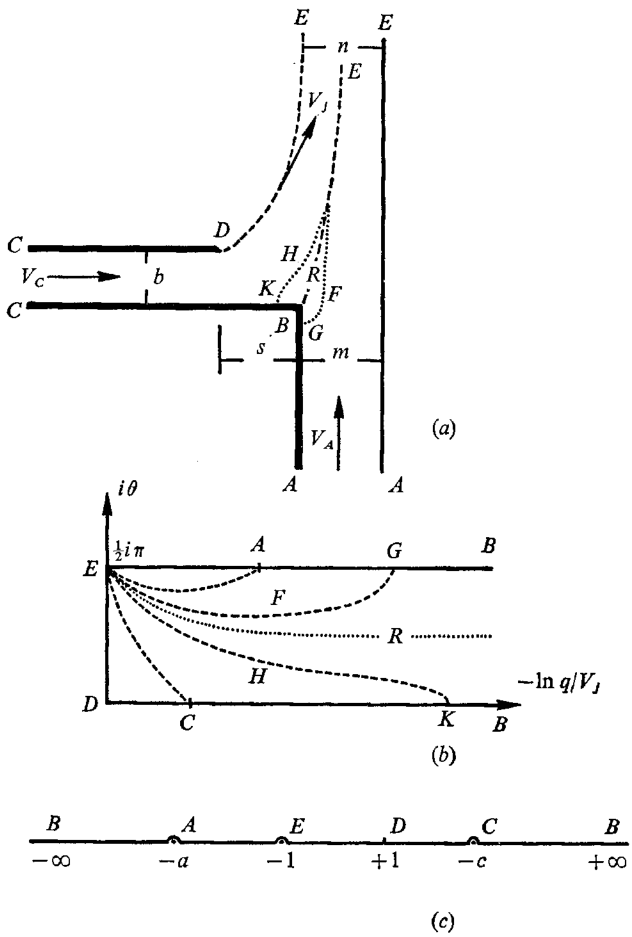

FIGURE 1. A typical combining flow and the transformation planes. (a) Physical plane, (b) Ω plane, (c) t plane.

a condition renders the solution unique. Its derivation for a general flow situation and application to the particular flow shown in figure 1 constitute the purpose of the paper.

2. Derivation of the stagnation condition

Consider a divided or combined flow with straight solid boundaries in the physical z plane. The use of the transformation

$$\Omega = -\ln q/V_j + i\theta, \tag{1}$$

where $q = |u+iv|$, θ the inclination of q measured from the x axis, and V_j the reference velocity, transforms the solid boundaries into horizontal lines and the free streamlines into vertical lines in the Ω plane. The stagnation point B is at infinity along the real axis.

The flow in the Ω plane may be transformed into either the upper or the lower half of another plane (called the t plane) through the use of the Schwarz–Christoffel transformation

$$\Omega = \int M \prod_{1}^{n} (t-t_r)^{-\alpha_r/\pi}\, dt + N. \tag{2}$$

As many factors $(t-t_r)$ are introduced into the transformation as there are vertices of the polygonal boundary in the Ω plane. The value of t_r is the locus of the transformed point on the real axis of the t plane, and α_r is the exterior angle of the polygon at that point in the Ω plane. Three of the t_r values can be assigned arbitrarily and the remainder must be given parametric values to be evaluated in terms of given quantities, along with M and N, from the integrated function. Points for which t_r is infinity need not be introduced no matter what the value of α_r.

The stagnation point which is at infinity in the Ω plane is ordinarily placed at $t = \pm\infty$ for the purpose of eliminating the corresponding $(t-t_r)$ factor in the Schwarz–Christoffel transformation. The advantages of such a choice are considerable and may often mean the difference between a closed-form solution and numerical integration of the $\Omega(t)$ function. Needless to say, the subsequent success with the analysis depends to a great extent on the degree of simplicity or complexity of the $\Omega(t)$ function, since the degree of difficulty of the various transformations depends in general on the number of discontinuities in the flow: sources, sinks, doublets, stagnation points, etc. The placing of the stagnation point at $t = \pm\infty$, however, does introduce a special problem and require that a separate stagnation condition be satisfied. In fact, it is the derivation of this special condition that prompted the present work.

The flow in the t plane is in general comprised of sources, sinks, and doublets with strengths Q_r and μ_k. In accordance with the sign convention used here, Q_r is taken positive for a sink and negative for a source. Then the potential function $w(t)$ may be written as

$$w(t) = \sum_{1}^{n} \mu_k (t-t_k)^{-1} + \sum_{1}^{m} (Q_r/\pi) \ln (t-t_r). \tag{3}$$

The velocity in the t plane is given by

$$dw/dt = -u + iv = -\sum_{1}^{n} \mu_k (t-t_k)^{-2} + \sum_{1}^{m} (Q_r/\pi)(t-t_r)^{-1}. \tag{4}$$

At $t = \pm\infty$, i.e. for any point so placed in the t plane, $dw/dt = 0$ automatically. In other words, the velocity at the stagnation point approaches zero regardless of the way the stagnation streamline ERB (figure 1 (a)) approaches the stagnation point B. Thus a condition has to be formulated for which the stagnation streamline bisects the wedge or the u and v components of velocity at the vertex of the wedge vanish simultaneously.

Rewriting (4) by noting that $\sum_{1}^{m} Q_r = 0$ or $Q_1 = -\sum_{2}^{m} Q_r$, one has

$$dw/dt = -(t-t_1)^{-1} \sum_{2}^{m} Q_r/\pi + \sum_{2}^{m} (Q_r/\pi)(t-t_r)^{-1} - \sum_{1}^{n} \mu_k (t-t_k)^{-2}. \tag{5}$$

Let us now assume that the point t_r corresponding to the stagnation point is located at a finite point $t = t_0$. Then the stagnation condition $(dw/dt)_{t=t_0} = 0$ reduces to

$$-(t_0-t_1)^{-1}\sum_2^m Q_r/\pi + \sum_2^m (Q_r/\pi)(t_0-t_r)^{-1} - \sum_1^n \mu_k(t_0-t_k)^{-2} = 0. \tag{6}$$

Simplifying, one has

$$\sum_2^m Q_r(t_r-t_1)/(t_0-t_1)(t_0-t_r) - \pi\sum_1^n \mu_k(t_0-t_k)^{-2} = 0. \tag{7}$$

Now multiplying with $(t_0-t_1)(t_0-t_p)$ where p is a particular value of r, (7) becomes

$$\sum_2^m Q_r(t_r-t_1)(t_0-t_p)/(t_0-t_r) - \pi\sum_1^n \mu_k(t_0-t_1)(t_0-t_p)/(t_0-t_k)^2 = 0. \tag{8}$$

Equation (8) would have been the necessary stagnation condition had the stagnation point been placed at $t = t_0 \mp \infty$. Now letting $t_0 \to \pm\infty$, one has

$$\sum_1^m Q_r(t_r-t_1) - \pi\sum_1^n \mu_k = 0, \tag{9}$$

or, by noting once again that $Q_1 = -\sum_2^m Q_r$, one finally has

$$\sum_1^m Q_r t_r - \pi\sum_1^n \mu_k = 0. \tag{10}$$

Equation (10) is the special stagnation condition sought. It should be noted that in the physical plane $Q_r = V_r h_r$ where V_r and h_r are the corresponding velocities and channel or asymptotic free-jet widths respectively. Consequently, (10), as written below, places an additional restriction on the flow rates and the location of the singularities in the t plane,

$$\sum_1^m V_r h_r t_r - \pi\sum_1^n \mu_k = 0. \tag{11}$$

Evidently, (10) could have also been derived by requiring that u and v become identical and vanish simultaneously as the common streamline approaches the stagnation point at the vertex of the wedge. Writing

$$t-t_r = \lambda_r \exp\{i\sigma_r\} \quad \text{and} \quad t-t_k = \lambda_k \exp\{i\sigma_k\}, \tag{12}$$

where λ and σ represent respectively the modulus and the argument of a complex vector for an arbitrary point P above the real axis of the t plane, inserting (12) in (4), separating the real and imaginary parts to find u and v, and finally writing $u = v$, one has

$$\sum_1^n (\mu_k/\lambda_k^2)(\cos 2\sigma_k - \sin 2\sigma_k) + \sum_1^m (Q_r\lambda_r/\pi\lambda_r^2)\sin\sigma_r - \sum_1^m (Q_r\lambda_r/\pi\lambda_r^2)\cos\sigma_r = 0. \tag{13}$$

Now multiplying (13) with λ_p^2, where p is a particular value of r, and noting that $Q_1 = -\sum_2^m Q_r$ and $\lambda_r \cos\sigma_r = t_1 - t_r$, (13) reduces to

$$\sum_1^n (\mu_k/\lambda_k^2)\,\lambda_p^2(\cos 2\sigma_k - \sin 2\sigma_k) + \sum_2^m [Q_r(\lambda_r - \lambda_1)\,\lambda_p^2/\lambda_r^2\pi]\sin\sigma_r - \sum_1^m (Q_r\lambda_p^2/\pi\lambda_r^2)\,(t_1 - t_r) = 0. \quad (14)$$

The second summation in (14) is zero, since $\lambda_r \sin\sigma_r = \lambda_1 \sin\sigma_1$ for a given point P in the t plane. Now letting $\lambda \to \infty$ and $\sigma \to \frac{1}{2}\pi$ in (14), as the point P approaches infinity or the commom streamline approaches the stagnation point, (14) reduces to (10). It is evident from the foregoing derivation that the flow on both sides of the wedge becomes identical in the immediate vicinity of the wedge and that the wedge angle ($\alpha = \frac{1}{2}\pi$ in the physical plane and π in the t plane) is bisected by the common streamline.

In the case of impinging free jets, there is no doublet, and $V_r = V_j =$ constant along all free surfaces. Thus, (11) reduces to

$$\sum_1^m h_r t_r = 0. \quad (15)$$

In this special case, the boundary of the hodograph plane is a circle (see Birkhoff & Zarantonello 1957, p. 48), i.e. $t_r = \exp\{i\alpha_r\}$, and (15) reduces to

$$\sum_1^m h_r \cos\alpha_r = \sum_1^m h_r \sin\alpha_r, \quad (16)$$

where α_r is the inclination of the rth jet with respect to the x axis and h_r the asymptotic width of the rth jet.

Equation (16) is equivalent to the condition that the stagnation point be located at the centre of the circle where there are no other singularities by virtue of the conservation of mass, i.e. $\Sigma h_r = 0$. Since (16) is also equivalent to the conservation of momentum, one ends up with $i+2$ equations between $i+3$ unknowns. Thus, contrary to expectation, the resulting flow is indeterminate, except in the case of parallel impinging jets (Birkhoff & Zarantonello 1957, p. 48). For the general flows considered herein, however, (11) is not equivalent to the conservation of momentum. In other words, it is a kinematic flow condition not derivable from dynamic considerations of energy and momentum. Hence, the problem is determinate and there is a unique relationship between the magnitude of combining or branching flows and the geometry of the system as long as no discontinuity is permitted in velocity across the common streamline. Finally, (11) permits one to determine the strength of the doublets to be placed in the t plane for flows combining with or bifurcating from an otherwise unbounded uniform flow.

3. Application of the stagnation condition to a jet penetration problem

The flow boundary considered herein consists of two normally impinging two-dimensional inviscid jets of finite extent (figure 1(a)), a vortex sheet of free streamline, and a common streamline extending from the stagnation point into the combined uniform flow. The determination of the contraction of the combined jets, free-streamline shape, pressure distribution, etc., constitutes the basis of the problem.

The pressure and hence the velocity along the external free streamline are assumed to remain constant. In real fluids, this boundary could be the outer surface of a separation bubble comprised of the same fluid, an entrainement interface between the deflected jet and the ambient fluid, or a surface of density and viscosity discontinuity if the physical characteristics of the jets are different. In any case, the complex problems associated with the determination of the pressure along the bubble boundary, entrainement of the ambient fluid into the jet, and of the reattachment of the combined flow to an adjacent boundary cannot yet be considered without introducing into the analysis a number of empirical parameters (Sarpkaya 1972*a*, pp. R2–1, R2–21).

The common streamline passing through the stagnation point will ordinarily be a line of velocity discontinuity and will give rise to mixing, shear-layer generated noise, and energy dissipation unless the stagnation condition derived above is satisfied. In other words, the conditions leading to a discontinuity surface must be analytically excluded from the inviscid flow analysis.

The aforementioned assumptions and the continuity relationship, together with the appropriate conformal transformations, suffice for the determination of the characteristics of the combined efflux.

If z and $w(z)$ are used for the complex variable and the complex potential, then the variable ζ defined as dw/dz is given by

$$\zeta = -u + iv = -q\exp\{-i\theta\}. \tag{17}$$

The flow region in the physical plane (figure 1 (*a*)) can be mapped onto a corresponding region in the Ω plane, as indicated in figure 1 (*b*), through

$$\Omega = \ln(-V_j/\zeta) = -\ln(q/V_j) + i\theta. \tag{18}$$

The polygonal boundary shown in figure 1 (*b*) may be transformed into the entire real axis of the t plane (figure 1 (*c*)), using the Schwarz–Christoffel transformation given by (2). The desired mapping function between Ω and t for the problem under consideration becomes

$$\Omega = M\int(t^2-1)^{-\frac{1}{2}}\,dt + N. \tag{19}$$

The evaluation of this integral and the use of the t_r values assigned to the points D and E in the t plane yield

$$\Omega = \tfrac{1}{2}ch^{-1}t. \tag{20}$$

The distances 'a' and 'c' shown in the t plane can be expressed as a function of the velocity ratios. To this end, the expression for Ω is introduced in (20) and the resulting equation evaluated at points A and C, so that

$$\left.\begin{aligned} a &= \tfrac{1}{2}[(V_j/V_A)^2 + (V_A/V_j)^2],\\ c &= \tfrac{1}{2}[(V_j/V_C)^2 + (V_C/V_j)^2],\end{aligned}\right\} \tag{21}$$

or

$$(V_j/V_C)^2 = c + (c^2-1)^{-\frac{1}{2}}, \quad (V_j/V_A)^2 = a + (a^2-1)^{-\frac{1}{2}}. \tag{22}$$

The complex potential $w(t)$ can be expressed in terms of t by the method of sources and sinks:

$$w(t) = [V_j n\ln(t+1) - V_A m\ln(t+a) - V_C b\ln(t-c)]/\pi. \tag{23}$$

Furthermore, t is related to θ along the free streamlines, where $\Omega = i\theta = \frac{1}{2}ch^{-1}t$, by

$$t = \cos 2\theta, \quad \cos\theta = [(1+t)/2]^{-\frac{1}{2}}, \quad \sin\theta = [(1-t)/2]^{-\frac{1}{2}}. \tag{24}$$

The appropriate distances in the physical plane and the contraction coefficient may be evaluated by noting that

$$n + \int_{t=1}^{t=-1} dx = m+s, \tag{25}$$

where the integral in (25) may be written, using (23) and (24), as

$$\int_{t=1}^{t=-1} dx = -\frac{1}{\pi V_j}\int_{1}^{-1}\left(\frac{1+t}{2}\right)^{\frac{1}{2}}\left[\frac{V_j n}{t+1} - \frac{mV_A}{t+a} - \frac{bV_C}{t-c}\right] dt. \tag{26}$$

Evaluating the integrals in (26), combining with (25), and rearranging, one finally has

$$\frac{1}{C_c} = \frac{m+s}{n} = \frac{\pi+2}{\pi} - \frac{V_C b}{V_j n}\bigg/(\pi\sqrt{2})\, F(a,c), \tag{27}$$

where C_c is defined by

$$C_c = n/(m+s), \tag{28}$$

and the function $F(a,c)$ is given by

$$F(a,c) = \frac{1+c}{a-1}\left(2\sqrt{2} - 2(a-1)^{\frac{1}{2}}\tan^{-1}\left(\frac{2}{a-1}\right)^{\frac{1}{2}}\right) + \left(2\sqrt{2} - (1+c)^{\frac{1}{2}}\ln\frac{(1+c)^{\frac{1}{2}}+\sqrt{2}}{(1+c)^{\frac{1}{2}}-\sqrt{2}}\right). \tag{29}$$

The premise of continuity, implicit in the above development, can be stated as

$$V_A m + V_C b = V_j n. \tag{30}$$

The parametric equations for the free streamline emanating from the point D may be obtained in a straightforward manner from

$$x - x_D = -\frac{1}{\pi V_j}\int_1^t \left(\frac{1+t}{2}\right)^{\frac{1}{2}}\frac{dw}{dt}dt \tag{31}$$

and

$$y - y_D = -\frac{1}{\pi V_j}\int_1^t \left(\frac{1-t}{2}\right)^{\frac{1}{2}}\frac{dw}{dt}dt, \tag{32}$$

through the use of (23).

Although (27)–(32) determine the characteristics of the flow field in terms of the parameters 'a' and 'c' or in terms of b/m, s/m, and V_C/V_A, the solution is not unique and is not necessarily physically applicable. The multiplicity of the solution stems, as previously discussed, from the fact that the common streamline can emanate from the stagnation point B in an infinite variety of directions while satisfying the condition that $\lim_{t\to\infty} dw/dt \to 0$, as seen from (23). The solution may be rendered unique and the discontinuity across the common streamline be eliminated by applying the stagnation condition given by (11) as

$$-V_C bc + V_A ma - V_j n = 0, \tag{33}$$

or, combining with the equation of continuity (30), one has

$$V_C b/V_A m = Q_R = Q_C/Q_A = (a-1)/(c+1). \tag{34}$$

Evidently, there is only one velocity or flow ratio for a given geometry (b/m and s/m) which satisfies the above condition.

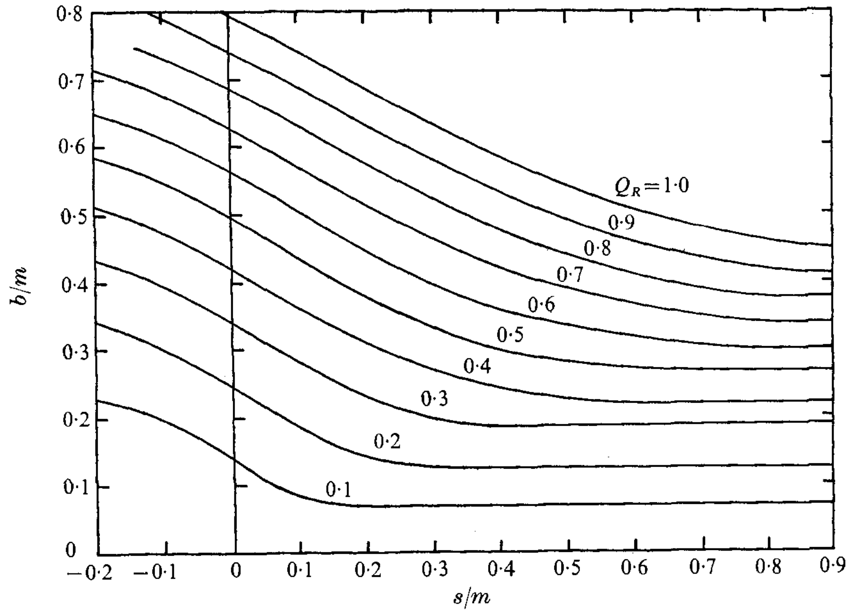

FIGURE 2. The jet geometry required for the elimination of the velocity discontinuity for various flow ratios.

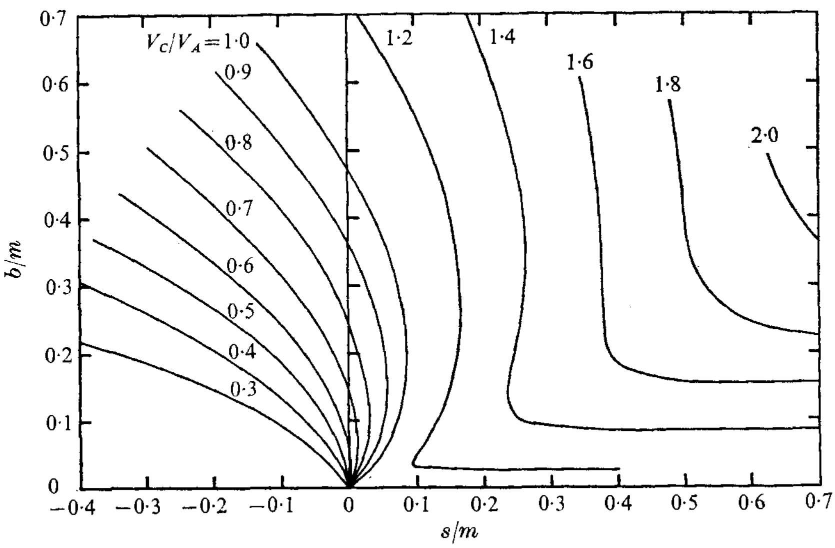

FIGURE 3. Velocity ratios for the flows and geometries shown in figure 2.

Results obtained by computer are presented in graphical form in figures 2–4. Figure 2 shows the variation of b/m with s/m for constant values of Q_R. The asymptotic values of b/m, obtained from (21), (22) and (34) are given by

$$(b/m)_{\min} = Q_R/\{(2Q_R+1)+[(2Q_R+1)^2-1]^{\frac{1}{2}}\}^{\frac{1}{2}}. \tag{35}$$

Figure 3 shows the variation of b/m with s/m for constant values of the velocity ratio V_C/V_A and figure 4 shows the variation of the contraction coefficient C_c with b/m for constant values of Q_R.

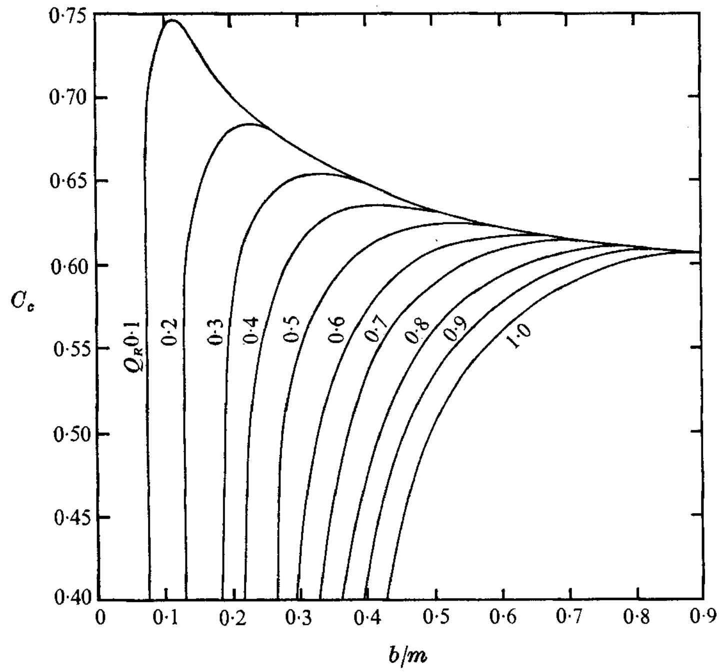

FIGURE 4. The contraction coefficient C_c against b/m for various values of Q_R.

A careful examination of figures 2–4 reveals several facts of major importance. First, the achievement of a smooth impingement with no velocity discontinuity requires the use of relatively small control nozzles (i.e. very small b/m values), and hence fairly large velocity ratios. Second, there is only one particular velocity or flow ratio for a given set of specific values of the jet geometry. Any variation in input-flow ratio from that predicted on the basis of no velocity discontinuity will result in the development of free-shear layers of varying intensity and in unwanted noise and energy dissipation in the flow. Finally, it is apparent from figure 4 that the contraction coefficient decreases, i.e. the acceleration of the combined jet increases, as b/m and/or Q_R increases.

4. Summary

A unique stagnation condition has been derived as a guide for analysing systems of combined or divided flows through the use of the free-streamline theory and the Schwarz–Christoffel transformation. The physical significance of such a condition has been illustrated through its application to the analysis of the penetration and deflexion of jets normal to a bounded stream. Additional applications are given in Sarpkaya (1972*b*) in a detailed study of the interaction of semi-confined turbulent jets.

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